$a_1=1,\ a_2=1$

$a_{n+2}=a_{n+1}+a_n \quad (n=1,2,3,\cdots)$

$\tan b_n=\dfrac{1}{a_n} \quad \left(0<b_n<\dfrac{\pi}{2}\right)$

(1) $\boxed{n\geqq 2 \text{ に対して、} a_{n+1}a_{n-1}-a_n^2 \text{ を求める}}$

$$C_n=a_{n+1}a_{n-1}-a_n^2$$

とすると

$$C_{n+1}=a_{n+2}a_n-a_{n+1}^2=(a_{n+1}+a_n)a_n-a_{n+1}(a_n+a_{n-1})=a_{n+1}\cancel{a_n}+a_n^2-\cancel{a_{n+1}a_n}-a_{n+1}a_{n-1}$$

$$=-\left(a_{n+1}a_{n-1}-a_n^2\right)$$

$$=-C_n$$

ここで $a_3=a_2+a_1=1+1=2$

$$C_2=a_3a_1-a_2^2=2\cdot1-1^2=1$$

よって $C_n=C_2(-1)^{n-2}=(-1)^{n-2}$

$$=\boxed{(-1)^n}\ //$$

(2) $\boxed{m\geqq 1 \text{ に対して } a_{2m}\cdot\tan\left(b_{2m+1}+b_{2m+2}\right) \text{ を求める}}$

$$a_{2m}\tan\left(b_{2m+1}+b_{2m+2}\right)=a_{2m}\cdot\frac{\tan b_{2m+1}+\tan b_{2m+2}}{1-\tan b_{2m+1}\,b_{2m+2}}=a_{2m}\cdot\frac{\frac{1}{a_{2m+1}}+\frac{1}{a_{2m+2}}}{1-\frac{1}{a_{2m+1}}\cdot\frac{1}{a_{2m+2}}}$$

$$=\frac{a_{2m}a_{2m+2}+a_{2m}a_{2m+1}}{a_{2m+1}a_{2m+2}-1}$$

ここで (1) で $n=2m+1$ として $a_{2m+2}a_{2m}-a_{2m+1}^2=(-1)^{2m+1}=-1$

$$\therefore\ a_{2m}a_{2m+2}=a_{2m+1}^2-1$$

$$(\text{分子})=a_{2m+1}^2-1+a_{2m}a_{2m+1}=a_{2m+1}\left(a_{2m+1}+a_{2m}\right)-1=a_{2m+1}a_{2m+2}-1 \quad (=\text{分母})$$

よって $a_{2m}\tan\left(b_{2m+1}+b_{2m+2}\right)=\boxed{1}\ //$

(3) $\boxed{\displaystyle\sum_{m=0}^{\infty}b_{2m+1}}$

(2) より $\tan\left(b_{2m+1}+b_{2m+2}\right)=\dfrac{1}{a_{2m}}=\tan b_{2m}$

$0<b_{2m+1}+b_{2m+2}<\pi$, $0<b_{2m}<\dfrac{\pi}{2}$ なので $b_{2m+1}+b_{2m+2}=b_{2m}$

$$\therefore\ b_{2m+1}=b_{2m}-b_{2m+2} \quad (m\geqq 1)$$

$N\geqq 1$ として

$$\sum_{m=0}^{N}b_{2m+1}=b_1+\sum_{m=1}^{N}\left(b_{2m}-b_{2m+2}\right)=b_1+b_2-b_{2N+2}$$

数列 $\{a_n\}$ の項は自然数で単調増加であるから $a_n\to\infty\ (n\to\infty)$ なので $\displaystyle\lim_{n\to\infty}\tan b_n=\lim_{n\to\infty}\frac{1}{a_n}=0$

$0<b_n<\dfrac{\pi}{2}$ であるから $\displaystyle\lim_{n\to\infty}b_n=0$

$\tan b_1=\dfrac{1}{a_1}=1$, $\tan b_2=\dfrac{1}{a_2}=1$ なので $b_1=\dfrac{\pi}{4}$, $b_2=\dfrac{\pi}{4}$

よって $\displaystyle\sum_{m=0}^{\infty}b_{2m+1}=\lim_{N\to\infty}\sum_{m=0}^{N}b_{2m+1}=\lim_{N\to\infty}\left(b_1+b_2-\underbrace{b_{2N+2}}_{0}\right)=\frac{\pi}{4}+\frac{\pi}{4}=\boxed{\frac{\pi}{2}}\ //$