$f(x) = x\log(1+x)$

(1) $\boxed{\int x\log(1+x)dx} = \dfrac{x^2-1}{2}\log(1+x) - \int \dfrac{(x-1)\cancel{(x+1)}}{2}\cdot\dfrac{1}{\cancel{1+x}}dx$ ← 部分積分法 $\left(\dfrac{x^2-1}{2}\right)' = x$ がよい

$= \boxed{\dfrac{x^2-1}{2}\log(1+x) - \dfrac{(x-1)^2}{4} + C \ (C\text{は積分定数})}$ //

(2) $y=f(x)\ (x\geqq 0)$ の逆関数を $y=g(x)\ (x\geqq 0)$

$g(a)=1,\ g(b)=2$

$\boxed{I = \int_a^b g(x)\,dx}$

$y=f(x) \iff x=g(y)\ (y\geqq 0)$

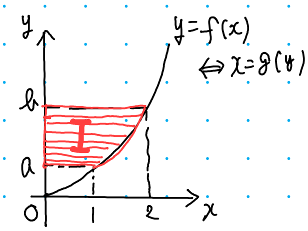
$I = \int_a^b g(y)\,dy = \int_1^2 x\dfrac{dy}{dx}dx$ （$x=g(y)$ と置換 $\begin{array}{c|cc} y & a & \to b \\ \hline x & 1 & 2 \end{array}$）

$= \left[x f(x)\right]_1^2 - \int_1^2 f(x)\,dx$ ← (1)

$= 2f(2) - f(1) - \left[\dfrac{x^2-1}{2}\log(1+x) - \dfrac{(x-1)^2}{4}\right]_1^2$

$= 2\cdot 2\log 3 - \log 2 - \left(\dfrac{3}{2}\log 3 - \dfrac{1}{4}\right)$

$= \boxed{\dfrac{5}{2}\log 3 - \log 2 + \dfrac{1}{4}}$ //



(3) $P(x) = \int_0^x \sqrt{1+f(t)}\,dt \quad (x\geqq 0)$

$y=P(x)$ の逆関数を $y=Q(x)\ (x\geqq 0)$

$\boxed{R(x) = \int_0^{P(x)} \dfrac{1}{Q'(v)}dv}$

$P(0)=0$

$P'(x) = \sqrt{1+f(x)}$

$u=Q(v) \iff v=P(u)$ と置換する

$\begin{array}{c|ccc} v & 0 & \to & P(x) \\ \hline u & 0 & \to & x \end{array}$ ← $0=P(0)$, $P(x)=P(u)$

$\dfrac{dv}{du} = P'(u) = \sqrt{1+f(u)}$ ← $v=P(u)$ を u で微分

$Q'(v) = \dfrac{du}{dv} = \dfrac{1}{\frac{dv}{du}} = \dfrac{1}{P'(u)} = \dfrac{1}{\sqrt{1+f(u)}}$ ← $u=Q(v)$ を v で微分

$R(x) = \int_0^x \sqrt{1+f(u)}\cdot\sqrt{1+f(u)}\,du = \int_0^x \{1+f(u)\}\,du = \left[u + \dfrac{u^2-1}{2}\log(1+u) - \dfrac{(u-1)^2}{4}\right]_0^x$

$= x + \dfrac{x^2-1}{2}\log(1+x) - \dfrac{(x-1)^2}{4} + \dfrac{1}{4}$

$= \boxed{\dfrac{x^2-1}{2}\log(1+x) - \dfrac{1}{4}x^2 + \dfrac{3}{2}x}$ //