極限 $\displaystyle\lim_{n\to\infty}\left\{\frac{(2n)!}{n!n^n}\right\}^{\frac{1}{n}}$ を求めよ．

〔2012　横国大 後期〕

〔解答例〕

$$\frac{(2n)!}{n!} = \frac{2n\cdot(2n-1)\cdot\dots\cdot(n+1)\cdot\cancel{n!}}{\cancel{n!}}$$

$$n^n = n\cdot n\cdot\ \cdots\ \cdot n$$

$a_n=\left\{\dfrac{(2n)!}{n!n^n}\right\}^{\frac{1}{n}}$ とおく．

右からの積に

$$\begin{aligned}\log a_n &= \log\left\{\frac{(2n)!}{n!n^n}\right\}^{\frac{1}{n}}\\ &= \frac{1}{n}\log\left(\frac{2n}{n}\cdot\frac{2n-1}{n}\cdot\dots\cdot\frac{n+1}{n}\cdot\frac{n!}{n!}\right) = \frac{1}{n}\log\left(\frac{n+1}{n}\cdot\frac{n+2}{n}\cdot\dots\cdot\frac{n+n}{n}\right)\\ &= \frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\dots+\log\left(1+\frac{n}{n}\right)\right\}\\ &= \frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{k}{n}\right)\end{aligned}$$

n個の和

これより

$$\begin{aligned}\lim_{n\to\infty}\log a_n &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{k}{n}\right)\\ &= \int_0^1\log(1+x)\,dx\\ &= \Big[(1+x)\log(1+x)-x\Big]_0^1\\ &= 2\log 2-1\\ &= \log\frac{4}{e}\end{aligned}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx$$

dx　$\int_0^1$　$f(x)$

よって，$\displaystyle\lim_{n\to\infty}a_n=\boldsymbol{\frac{4}{e}}$

$\displaystyle\lim_{n\to\infty}\log a_n=\log\alpha$

ならば

$\displaystyle\lim_{n\to\infty}a_n=\alpha$