[1] 微分可能な関数 $f(x)$, $g(x)$ について

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

← 部分積分法

が成り立つことを説明せよ.

〔解答例〕

積の微分法より $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$

すなわち $f'(x)g(x) = \{f(x)g(x)\}' - f(x)g'(x)$

これを積分すると

$$\int f'(x)g(x)\,dx = \int \left(\{f(x)g(x)\}' - f(x)g'(x)\right)dx$$

$$= f(x)g(x) - \int f(x)g'(x)\,dx$$

積分

微分

$\int \log x\,dx = x\log x - x + C$

(px+q)とせず定数 q はさりげなく消しました

[2] $$\int \log(px+q)\,dx = \frac{1}{p}\{(px+q)\log(px+q) - px\} + C$$

$$= \left(\boldsymbol{x} + \frac{\boldsymbol{q}}{\boldsymbol{p}}\right)\mathbf{log}\,(\boldsymbol{px}+\boldsymbol{q}) - \boldsymbol{x} + \boldsymbol{C}$$ （C は積分定数）

3～7 は 瞬間部分積分で!

別 $$\int \log(px+q)\,dx = \frac{px+q}{p}\log(px+q) - \int \frac{px+q}{p}\cdot\frac{p}{px+q}\,dx$$

$$= \left(\boldsymbol{x} + \frac{\boldsymbol{q}}{\boldsymbol{p}}\right)\mathbf{log}\,(\boldsymbol{px}+\boldsymbol{q}) - \boldsymbol{x} + \boldsymbol{C}$$ （C は積分定数）

[3] $$\int_0^1 x^3e^x\,dx = \Big[e^x(x^3-3x^2+6x-6)\Big]_0^1 = -2e-(-6) = \boxed{-2e+6}$$

+	x^3	e^x
−	$3x^2$	e^x
+	$6x$	e^x
−	6	e^x

[4] $$\int_0^1 x^3e^{-x}\,dx = \Big[-e^{-x}(x^3+3x^2+6x+6)\Big]_0^1 = -e^{-1}\cdot 16+6 = \boxed{6-\frac{16}{e}}$$

+	x^3	$-e^{-x}$
−	$3x^2$	e^{-x}
+	$6x$	$-e^{-x}$
−	6	e^{-x}

[5] $$\int_0^\pi x^3\sin x\,dx = \Big[x^3(-\cos x)+3x^2\sin x+6x\cos x+6(-\sin x)\Big]_0^\pi$$

$$= \boxed{\pi^3-6\pi}$$

+	x^3	$-\cos x$
−	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
−	6	$\sin x$

[6] $$\int_0^\pi x^3\cos x\,dx = \Big[x^3(\sin x)+3x^2\cos x+6x(-\sin x)+6(-\cos x)\Big]_0^\pi$$

$$= \boxed{-3\pi^2+12}$$

+	x^3	$\sin x$
−	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
−	6	$\cos x$

[7] $$\int_0^\pi x^3\cos 2x\,dx = \left[\frac{x^3}{2}\sin 2x+\frac{3}{4}x^2\cos 2x-\frac{3}{4}x\sin 2x-\frac{3}{8}\cos 2x\right]_0^\pi$$

$$= \boxed{\frac{3}{4}\pi^2}$$

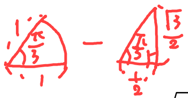
+	x^3	$\frac{\sin 2x}{2}$
−	$3x^2$	$-\frac{\cos 2x}{4}$
+	$6x$	$-\frac{\sin 2x}{8}$
−	6	$\frac{\cos 2x}{16}$

$\boxed{8}$ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{3+\cos^2 x}\,dx = \int_0^{\frac{\pi}{2}} \left\{-\frac{-2\sin x\cos x}{3+\cos^2 x}\,dx\right\} = \Big[-\log|3+\cos^2 x|\Big]_0^{\frac{\pi}{2}}$

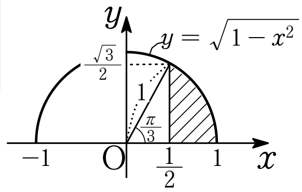
$$= -\log 3 + \log 4 = \boxed{\log\frac{4}{3}}$$

$\displaystyle\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + C$

$\boxed{9}$ 右図の斜線部の面積を考えて



$$\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\cdot 1^2 \cdot \frac{\pi}{3} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \boxed{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$$



別 $x = \sin\theta$ とおくと $\dfrac{dx}{d\theta} = \cos\theta$, $\begin{array}{c||ccc} x & \frac{1}{2} & \to & 1 \\ \hline \theta & \frac{\pi}{6} & \to & \frac{\pi}{2} \end{array}$

置換してもできます

$$\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cdot\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta = \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$$

$x=\sqrt{3}\tan\theta$ と置換 $\begin{array}{c|ccc} x & 0 & \to & 1 \\ \hline \theta & 0 & \to & \frac{\pi}{6} \end{array}$

$\boxed{10}$ $\displaystyle\int_0^1 \frac{1}{3+x^2}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{3+3\tan^2\theta}\cdot\frac{dx}{d\theta}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{\cos^2\theta}{3}\cdot\frac{\sqrt{3}}{\cos^2\theta}\,d\theta$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{3}}\,d\theta = \left[\frac{\theta}{\sqrt{3}}\right]_0^{\frac{\pi}{6}} = \boxed{\frac{\pi}{6\sqrt{3}}}$$

$\dfrac{1}{a^2+x^2}$ の形は $x=a\tan\theta$ の置換
$(a>0)$ $\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$

$\boxed{11}$ $x = \sin\theta$ とおくと $\dfrac{dx}{d\theta} = \cos\theta$, $\begin{array}{c||ccc} x & 0 & \to & 1 \\ \hline \theta & 0 & \to & \frac{\pi}{2} \end{array}$

$\sqrt{a^2-x^2}$ の形は $x=a\sin\theta$ の置換
$(a>0)$ $\left(-\frac{\pi}{2}\leqq\theta\leqq\frac{\pi}{2}\right)$

$$I = \int_0^1 x^2\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta}\,\frac{dx}{d\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$$

ここで $\sin^2\theta\cos^2\theta = (\sin\theta\cos\theta)^2 = \left(\dfrac{\sin 2\theta}{2}\right)^2 = \dfrac{\sin^2 2\theta}{4} = \dfrac{1-\cos 4\theta}{8}$

積分できる形に

$$I = \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{8}\,d\theta = \left[\frac{\theta}{8} - \frac{\sin 4\theta}{32}\right]_0^{\frac{\pi}{2}} = \boxed{\frac{\pi}{16}}$$

$\boxed{12}$ 部分積分法を用いて

$\log(x^2+1)$ を微分する形を考えておく

$$\int \underbrace{x^3}_{積分} \log(x^2+1)\,dx = \frac{x^4-1}{4}\log(x^2+1) - \int \frac{x^4-1}{4}\cdot\underbrace{\frac{2x}{x^2+1}}_{微分}\,dx$$

$$= \frac{x^4-1}{4}\log(x^2+1) - \int \frac{(x^2-1)(x^2+1)\cdot 2x}{4(x^2+1)}\,dx$$

x^2+1 を約分

$$= \frac{x^4-1}{4}\log(x^2+1) - \int\left(\frac{x^3}{2} - \frac{x}{2}\right)dx$$

$$= \frac{x^4-1}{4}\log(x^2+1) - \frac{x^4}{8} + \frac{x^2}{4} + C \quad (C \text{ は積分定数})$$

$$\int_0^1 x^3\log(x^2+1)\,dx = \left[\frac{x^4-1}{4}\log(x^2+1) - \frac{x^4}{8} + \frac{x^2}{4}\right]_0^1 = -\frac{1}{8} + \frac{1}{4}$$

$$= \boxed{\frac{1}{8}}$$

13

$$\frac{1}{\cos x}=\frac{\cos x}{\cos^2 x}=\frac{\cos x}{(1-\sin x)(1+\sin x)}=\frac{1}{2}\left(\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}\right)$$

$$\int_0^{\frac{\pi}{3}}\frac{1}{\cos x}dx=\frac{1}{2}\int_0^{\frac{\pi}{3}}\left(\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}\right)dx$$

↑ $\frac{1}{\sin x}$ と同じ

$$=\frac{1}{2}\Big[-\log(1-\sin x)+\log(1+\sin x)\Big]_0^{\frac{\pi}{3}}=\frac{1}{2}\left[\log\frac{1+\sin x}{1-\sin x}\right]_0^{\frac{\pi}{3}}$$

$$=\frac{1}{2}\log\frac{2+\sqrt{3}}{2-\sqrt{3}}=\frac{1}{2}\log(2+\sqrt{3})^2=\boxed{\log(2+\sqrt{3})}$$

14　$1\leqq x\leqq 4$ において　$\log(x^2)=2\log|x|=2\log x$

14, 15 は京大の過去問
(京大は部分積分が好き!?)

部分積分法を用いて

$$\int\sqrt{x}\log(x^2)\,dx=\int 2x^{\frac{1}{2}}\log x\,dx=\frac{4}{3}x^{\frac{3}{2}}\log x-\int\frac{4}{3}x^{\frac{3}{2}}\cdot\frac{1}{x}\,dx$$

$$=\frac{4}{3}x^{\frac{3}{2}}\log x-\int\frac{4}{3}x^{\frac{1}{2}}\,dx$$

$$=\frac{4}{3}x^{\frac{3}{2}}\log x-\frac{8}{9}x^{\frac{3}{2}}+C\quad(C\text{ は積分定数})$$

よって

$$\int_1^4\sqrt{x}\log(x^2)\,dx=\left[\frac{4}{3}x^{\frac{3}{2}}\log x-\frac{8}{9}x^{\frac{3}{2}}\right]_1^4=\frac{4}{3}\cdot 8\log 4-\frac{8}{9}(8-1)$$

$$=\boxed{\frac{64}{3}\log 2-\frac{56}{9}}$$

15　部分積分法を用いて

$\int\tan x\,dx=\int\frac{\sin x}{\cos x}dx=-\log|\cos x|+C$

$$\int\frac{x}{\cos^2 x}dx=x\tan x-\int\tan x\,dx=x\tan x+\log|\cos x|+C\quad(C\text{ は積分定数})$$

$$\int_0^{\frac{\pi}{4}}\frac{x}{\cos^2 x}dx=\Big[x\tan x+\log|\cos x|\Big]_0^{\frac{\pi}{4}}=\frac{\pi}{4}+\log\frac{1}{\sqrt{2}}$$

$$=\boxed{\frac{\pi}{4}-\frac{1}{2}\log 2}$$

定積分の計算は他にもいろいろありますが
とりあえず今回はここまでにします