$I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\, dx$ $(n = 0, 1, 2, \cdots)$ とおく．

(1) I_0，I_1，I_2，I_3 をそれぞれ求めよ．

(2) I_{n+2} を n，I_n を用いて表せ．

(3) I_4，I_5 をそれぞれ求めよ．

〔 解答例 〕

$\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)|$

(1) $I_0 = \displaystyle\int_0^{\frac{\pi}{4}} dx = \boldsymbol{\frac{\pi}{4}}$

$$I_1 = \int_0^{\frac{\pi}{4}} \tan x\, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\, dx = \Big[-\log|\cos x| \Big]_0^{\frac{\pi}{4}} = -\log\frac{1}{\sqrt{2}}$$

$$= \boldsymbol{\frac{1}{2}\log 2}$$

$(\tan x)' = \frac{1}{\cos^2 x}$

$$I_2 = \int_0^{\frac{\pi}{4}} \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - 1\right) dx = \Big[\tan x - x \Big]_0^{\frac{\pi}{4}} = \boldsymbol{1 - \frac{\pi}{4}}$$

$$I_3 = \int_0^{\frac{\pi}{4}} \tan^3 x\, dx = \int_0^{\frac{\pi}{4}} \tan x \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} \tan x\left(\frac{1}{\cos^2 x} - 1\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x}\cdot\tan x - \tan x\right) dx = \Big[\frac{1}{2}\tan^2 x \Big]_0^{\frac{\pi}{4}} - I_1$$

$$= \boldsymbol{\frac{1}{2} - \frac{1}{2}\log 2}$$

$\int g'(x)\{g(x)\}^{\alpha}\,dx = \frac{1}{\alpha+1}\{g(x)\}^{\alpha+1} + C$ $(\alpha \neq -1)$

上の I_3 と同様に変形できる

(2) $$I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^{n+2} x\, dx = \int_0^{\frac{\pi}{4}} \tan^n x \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} \tan^n x\left(\frac{1}{\cos^2 x} - 1\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x}\cdot\tan^n x - \tan^n x\right) dx = \Big[\frac{1}{n+1}\tan^{n+1} x \Big]_0^{\frac{\pi}{4}} - I_n$$

よって $\boldsymbol{I_{n+2} = \frac{1}{n+1} - I_n}$

部分積分法を用いない積分の漸化式

(3) $$I_4 = \frac{1}{3} - I_2 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \boldsymbol{\frac{\pi}{4} - \frac{2}{3}}$$

$$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\log 2\right) = \boldsymbol{\frac{1}{2}\log 2 - \frac{1}{4}}$$