$m,\ n$ を 0 以上の整数とする．

$$B(m,\ n)=\int_\alpha^\beta (x-\alpha)^m(x-\beta)^n\,dx$$

について，次の問に答えよ．

(1)　$B(m+n,\ 0)$ を求めよ．

(2)　$n \geqq 1$ のとき，$B(m,\ n)$ を $B(m+1,\ n-1)$ で表せ．

(3)　$B(m,\ n)$ を求めよ．

〔 解答例 〕

(1)
$$\begin{aligned} B(m+n,\ 0) &= \int_\alpha^\beta (x-\alpha)^{m+n}\,dx \\ &= \left[\frac{1}{m+n+1}(x-\alpha)^{m+n+1}\right]_\alpha^\beta = \boldsymbol{\frac{1}{m+n+1}(\beta-\alpha)^{m+n+1}} \end{aligned}$$

(2)　部分積分法を用いて

$$\begin{aligned} B(m,\ n) &= \int_\alpha^\beta (x-\alpha)^m(x-\beta)^n\,dx \\ &= \left[\frac{1}{m+1}(x-\alpha)^{m+1}(x-\beta)^n\right]_\alpha^\beta - \int_\alpha^\beta \frac{1}{m+1}(x-\alpha)^{m+1}\cdot n(x-\beta)^{n-1}\,dx \\ &= -\frac{n}{m+1}\int_\alpha^\beta (x-\alpha)^{m+1}(x-\beta)^{n-1}\,dx \end{aligned}$$

よって　$\boldsymbol{B(m,\ n)=-\frac{n}{m+1}B(m+1,\ n-1)}$　……㊟

(3)　㊟ を繰り返して

$$\begin{aligned} B(m,\ n) &= \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+2}\right)B(m+2,\ n-2) \\ &= \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+2}\right)\cdots\cdots\left(-\frac{1}{m+n}\right)B(m+n,\ 0) \quad \times\frac{m!}{m!} \\ &= (-1)^n\frac{m!n!}{(m+n)!}\frac{1}{m+n+1}(\beta-\alpha)^{m+n+1} \quad (\because (1)) \\ &= \boldsymbol{(-1)^n\frac{m!n!}{(m+n+1)!}(\beta-\alpha)^{m+n+1}} \end{aligned}$$

〔 補足 〕

$$B(1,\ 1)=\int_\alpha^\beta (x-\alpha)(x-\beta)\,dx=(-1)\frac{1!1!}{3!}(\beta-\alpha)^3=-\frac{1}{6}(\beta-\alpha)^3$$

$$B(1,\ 2)=\int_\alpha^\beta (x-\alpha)(x-\beta)^2\,dx=(-1)^2\frac{1!2!}{4!}(\beta-\alpha)^4=\frac{1}{12}(\beta-\alpha)^4$$

$$B(2,\ 1)=\int_\alpha^\beta (x-\alpha)^2(x-\beta)\,dx=(-1)\frac{1!2!}{4!}(\beta-\alpha)^4=-\frac{1}{12}(\beta-\alpha)^4$$

$$B(2,\ 2)=\int_\alpha^\beta (x-\alpha)^2(x-\beta)^2\,dx=(-1)^2\frac{2!2!}{5!}(\beta-\alpha)^5=\frac{1}{30}(\beta-\alpha)^5$$

よく使う積分なので
覚えておくとよい