次の定積分を求めよ．

(1)　$I=\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx$

(2)　$J=\displaystyle\int_0^1 x^3\log(x^2+1)\,dx$

〔2021　神大 理系 前期〕

〔解答例〕

(1)　$x=\sin\theta$ とおくと $\dfrac{dx}{d\theta}=\cos\theta$,
$\begin{array}{c||ccc} x & 0 & \to & 1 \\ \hline \theta & 0 & \to & \frac{\pi}{2} \end{array}$

置換積分

$$I=\int_0^1 x^2\sqrt{1-x^2}\,dx=\int_0^{\frac{\pi}{2}}\sin^2\theta\sqrt{1-\sin^2\theta}\,\frac{dx}{d\theta}\,d\theta=\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta$$

ここで $\sin^2\theta\cos^2\theta=(\sin\theta\cos\theta)^2=\left(\dfrac{\sin 2\theta}{2}\right)^2=\dfrac{\sin^2 2\theta}{4}=\dfrac{1-\cos 4\theta}{8}$

$$I=\int_0^{\frac{\pi}{2}}\frac{1-\cos 4\theta}{8}\,d\theta=\left[\frac{\theta}{8}-\frac{\sin 4\theta}{32}\right]_0^{\frac{\pi}{2}}=\boldsymbol{\frac{\pi}{16}}$$

$\sin\alpha\cos\alpha=\dfrac{\sin 2\alpha}{2}$

$\sin^2\alpha=\dfrac{1-\cos 2\alpha}{2}$

(2)　部分積分法を用いて

$$\begin{aligned}\int x^3\log(x^2+1)\,dx&=\frac{x^4-1}{4}\log(x^2+1)-\int\frac{x^4-1}{4}\cdot\frac{2x}{x^2+1}\,dx\\&=\frac{x^4-1}{4}\log(x^2+1)-\int\frac{x^4-1}{4}\cdot\frac{2x}{x^2+1}\,dx\\&=\frac{x^4-1}{4}\log(x^2+1)-\int\frac{(x^2-1)(x^2+1)\cdot 2x}{4(x^2+1)}\,dx\\&=\frac{x^4-1}{4}\log(x^2+1)-\int\left(\frac{x^3}{2}-\frac{x}{2}\right)dx\\&=\frac{x^4-1}{4}\log(x^2+1)-\frac{x^4}{8}+\frac{x^2}{4}+C\ \ (C\text{ は積分定数})\end{aligned}$$

㋝　㋒

$\left(\dfrac{x^4-1}{4}\right)'=x^3$

だと計算がラク

後ろの積分で x^2+1 を消せる!

よって

$$\begin{aligned}J&=\int_0^1 x^3\log(x^2+1)\,dx\\&=\left[\frac{x^4-1}{4}\log(x^2+1)-\frac{x^4}{8}+\frac{x^2}{4}\right]_0^1\\&=-\frac{1}{8}+\frac{1}{4}\\&=\boldsymbol{\frac{1}{8}}\end{aligned}$$

別
$$\begin{aligned}\int x^3\log(x^2+1)\,dx&=\frac{x^4}{4}\log(x^2+1)-\int\frac{x^4}{4}\cdot\frac{2x}{x^2+1}\,dx\\&=\frac{x^4}{4}\log(x^2+1)-\int\frac{x^5}{2(x^2+1)}\,dx\\&=\frac{x^4}{4}\log(x^2+1)-\frac{1}{2}\int\frac{(x^2+1)(x^3-x)+x}{(x^2+1)}\,dx\\&=\frac{x^4}{4}\log(x^2+1)-\frac{1}{2}\int\left(x^3-x+\frac{x}{x^2+1}\right)dx\\&=\frac{x^4}{4}\log(x^2+1)-\frac{1}{2}\left\{\frac{x^4}{4}-\frac{x^2}{2}+\frac{1}{2}\log(x^2+1)\right\}\\&=\frac{x^4-1}{4}\log(x^2+1)-\frac{x^4}{8}+\frac{x^2}{4}+C\ \ (C\text{ は積分定数})\end{aligned}$$

㋝　㋒

$\left(\dfrac{x^4}{4}\right)'=x^3$

だと激しい

念のため，やっておきます!

分子を分母でわる

$$\begin{array}{r} x^3-x \\ x^2+1\,\big)\,\overline{x^5} \\ \underline{x^5+x^3} \\ -x^3 \\ \underline{-x^3-x} \\ x \end{array}$$