次の定積分の値を求めよ．

(1) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x}\,dx$ (2) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x}$

〔2019 京大 理系 前期〕

〔解答例〕

$(\tan x)' = \frac{1}{\cos^2 x}$ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\log|\cos x|$

(1) 部分積分法を用いて

$$\int \frac{x}{\cos^2 x}\,dx = x\tan x - \int \tan x\,dx = x\tan x + \log|\cos x| + C \quad (C\text{ は積分定数})$$

$$\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x}\,dx = \Big[\, x\tan x + \log|\cos x|\,\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \log\frac{1}{\sqrt{2}}$$

$$= \boldsymbol{\frac{\pi}{4} - \frac{1}{2}\log 2}$$

(2)
$$\frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{(1-\sin x)(1+\sin x)} = \frac{1}{2}\left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)dx$$

$\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + C$

$$= \frac{1}{2}\Big[\, -\log|1-\sin x| + \log|1+\sin x|\,\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\,\log\left|\frac{1+\sin x}{1-\sin x}\right|\,\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{1}{2}\log(\sqrt{2}+1)^2$$

$$= \boldsymbol{\log(\sqrt{2}+1)}$$