

次の定積分の値を求めよ.

$$(1) \int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx$$

$$(2) \int_0^{\frac{\pi}{4}} \frac{dx}{\cos x}$$

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[解答例]

$$(1) \text{ 部分積分法を用いて} \quad \begin{aligned} (\tan x)' &= \frac{1}{\cos^2 x} \\ \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = -\log |\cos x| \end{aligned}$$

$$\int \frac{x}{\cos^2 x} dx = x \tan x - \int \tan x \, dx = x \tan x + \log |\cos x| + C \quad (C \text{ は積分定数})$$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx &= \left[x \tan x + \log |\cos x| \right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \log \frac{1}{\sqrt{2}} \\ &= \frac{\pi}{4} - \frac{1}{2} \log 2 \end{aligned}$$

$$(2) \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{(1-\sin x)(1+\sin x)} = \frac{1}{2} \left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x} \right)$$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x} \right) dx \quad \text{② } \int \frac{g'(x)}{g(x)} dx = \log |g(x)| + C \\ &= \frac{1}{2} \left[-\log |1-\sin x| + \log |1+\sin x| \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} \left[\log \left| \frac{1+\sin x}{1-\sin x} \right| \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} \log \frac{\sqrt{2}+1}{\sqrt{2}-1} \\ &= \frac{1}{2} \log (\sqrt{2}+1)^2 \\ &= \log(\sqrt{2}+1) \end{aligned}$$