次の定積分を求めよ．

（1）　$I=\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx$

（2）　$J=\displaystyle\int_0^1 x^3\log(x^2+1)\,dx$

〔2021　神大 理系 前期〕

〔解答例〕

$\sqrt{a^2-x^2}$ の形は $x=a\sin\theta$ $\left(-\frac{\pi}{2}\leqq\theta\leqq\frac{\pi}{2}\right)$ $(a>0)$ と置換するのが定石！

（1）　$x=\sin\theta$ とおくと $\dfrac{dx}{d\theta}=\cos\theta$，

x	$0\to 1$
θ	$0\to\frac{\pi}{2}$

$$I=\int_0^1 x^2\sqrt{1-x^2}\,dx=\int_0^{\frac{\pi}{2}}\sin^2\theta\sqrt{1-\sin^2\theta}\,\frac{dx}{d\theta}d\theta=\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta$$

ここで $\sin^2\theta\cos^2\theta=(\sin\theta\cos\theta)^2=\left(\dfrac{\sin2\theta}{2}\right)^2=\dfrac{\sin^2 2\theta}{4}=\dfrac{1-\cos4\theta}{8}$

$$I=\int_0^{\frac{\pi}{2}}\frac{1-\cos4\theta}{8}\,d\theta=\left[\frac{1}{8}\theta-\frac{\sin4\theta}{32}\right]_0^{\frac{\pi}{2}}=\boldsymbol{\frac{\pi}{16}}$$

$\left(\dfrac{x^4-1}{4}\right)'=x^3$

$\left(\dfrac{x^4}{4}\right)'=x^3$ は㊊

（2）　$J=\displaystyle\int_0^1 \underbrace{x^3}_{\text{↓積分}}\log(x^2+1)\,dx$

$$=\left[\frac{x^4-1}{4}\log(x^2+1)\right]_0^1-\int_0^1\frac{x^4-1}{4}\cdot\underbrace{\frac{2x}{x^2+1}}_{\text{微分}}dx\quad(\because\ 部分積分法)$$

$$=-\int_0^1\frac{(x^2-1)(x^2+1)\cdot2x}{4(x^2+1)}dx$$

$$=-\int_0^1\left(\frac{x^3}{2}-\frac{x}{2}\right)dx=-\left[\frac{x^4}{8}-\frac{x^2}{4}\right]_0^1=-\left(\frac{1}{8}-\frac{1}{4}\right)$$

$$=\boldsymbol{\frac{1}{8}}$$

㊊ $\displaystyle\int \underbrace{x^3}_{\text{積分}}\log(x^2+1)\,dx=\frac{x^4}{4}\log(x^2+1)-\int\frac{x^4}{4}\cdot\underbrace{\frac{2x}{x^2+1}}_{\text{微分}}dx$　（∵ 部分積分法）

$$=\frac{x^4}{4}\log(x^2+1)-\int\frac{x^5}{2(x^2+1)}dx$$

$$\begin{array}{r} x^3-x \\ x^2+1\,\overline{)\,x^5\qquad\quad} \\ \underline{x^5+x^3} \\ -x^3 \\ \underline{-x^3-x} \\ x \end{array}$$

$$=\frac{x^4}{4}\log(x^2+1)-\frac{1}{2}\int\frac{(x^2+1)(x^3-x)+x}{(x^2+1)}dx$$

$$=\frac{x^4}{4}\log(x^2+1)-\frac{1}{2}\int\left(x^3-x+\frac{x}{x^2+1}\right)dx$$

$$=\frac{x^4}{4}\log(x^2+1)-\frac{1}{2}\left\{\frac{x^4}{4}-\frac{x^2}{2}+\frac{1}{2}\log(x^2+1)\right\}$$

$$=\frac{x^4-1}{4}\log(x^2+1)-\frac{x^4}{8}+\frac{x^2}{4}+C\quad(C\ は積分定数)$$

よって

$$J=\int_0^1 x^3\log(x^2+1)\,dx$$

$$=\left[\frac{x^4-1}{4}\log(x^2+1)-\frac{x^4}{8}+\frac{x^2}{4}\right]_0^1$$

$$=-\frac{1}{8}+\frac{1}{4}$$

$$=\boldsymbol{\frac{1}{8}}$$